Interpreting rates of change in applied context: reflecting on students reasoning

Workshop presenters:
Tatiana Sango tatiana.sango@uct.ac.za
Sanet Steyn sanet.steyn@uct.ac.za

The purpose of this workshop is to look at the strategies of understanding and developing learners' reasoning when solving problems with rates of change. The rate of change concept is one of the most important applications of derivatives. We use the fact that $f'(x)$ represents the rate of change of $f(x)$.

Rates of change: calculus of motion

Key words
acceleration
average rate of change
instantaneous rate of change of f with respect to x at a point
position function
speed
distance
tangent line
displacement
velocity

Example 1: Average vs instantaneous rate of change
The distance an object travels over time *t* is given by the function $s = 5 - 3t^2$, where *s* is measured in metres and *t* is measured in seconds.
a) What is the average rate of change of the distance in the interval [0, 2]?

The average rate of change is given by the gradient ratio: $\frac{change\ in\ distance}{change\ in\ time} = \frac{s(2)-s(0)}{2-0}$

$$\frac{(5-3(2)^2)-(5-3(0)^2)}{2-0} = \frac{(5-12)-5}{2} = -6$$

The rate of change in the distance *s* is -6 m/s.
b) What is the instantaneous rate of change when *t* = 3?

To find the instantaneous rate of change of the distance, first find the derivative of the function. Then find the value of s at *t* = 3.

$$s' = -6t$$

If *t* = 3, then $s' = -6(3) = -18$.

The instantaneous rate of change in the distance at *t* = 3 seconds is -18 m/s.

Example 2: Velocity and acceleration
The function describing the position of an object dropped from a height of 100 meters is $s = 100 - 4{,}7t^2$, where s is measured in meters and t is measured in seconds. Find the object's velocity and acceleration when it hits the ground.

The velocity is the first derivative of position with respect to time:

$$v(t) = s'(t) = -9{,}4t$$

The acceleration is the first derivative of velocity with respect to time:

$$a(t) = v'(t) = -9{,}4$$

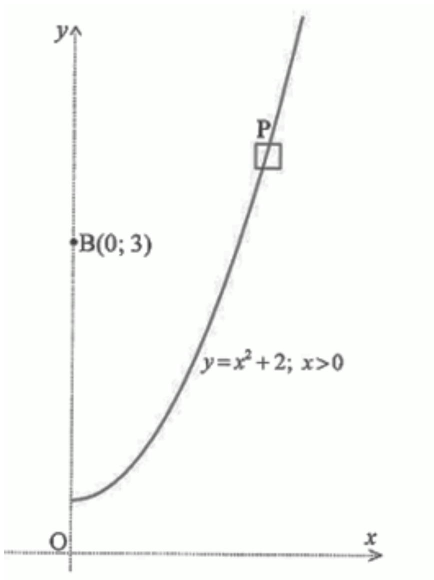
To find the velocity and acceleration of the object when it hits the ground, we first find the time at which it hits the ground by setting $s(t)$ equal to 0 and solving for t.

$$100 - 4{,}7t^2 = 0$$
$$4{,}7t^2 = 100$$
$$t^2 = \frac{100}{4{,}7}$$
$$t = \sqrt{\frac{100}{4{,}7}} \approx 4{,}6$$

The object hits the ground at approximately 4,6 seconds.
Finally, evaluate velocity when the object hits the ground:
$v(4{,}6) = -9{,}4(4{,}6) = -43{,}24$ m/s
The negative sign indicates that the velocity is directed downward. The object's acceleration is the same as it is for all values of t: approximately -9,4 m/s^2.
The acceleration is a constant (due to gravity), and is directed downward (negative sign).

2017 November examination

Question 9
An aerial view of a stretch of road is shown in the diagram below. The road can be described by the function $y = x^2 + 2, x \geq 0$ if the coordinate axes (dotted lines) are chosen as shown in the diagram.



Benny sits at a vantage point B (0; 3) and observes a car, P, travelling along the road.

Calculate the distance between Benny and the car, when the car is closest to Benny.

Solution with teaching and learning steps

Step 1: Construct learners' understanding of the problem (extrapolating information/inferencing; developing reasoning)

- Benny is not moving; his position is fixed at point B (0;3)
- Curve represents the road
- Car is moving, its position is described by the point P, "sliding" on the curve (travelling along the road)
- The coordinates of point P are changing, but the point remains in 1st quadrant, where $x \geq 0$ and $y \geq 0$
- The distance between Benny and the car is not always the same because the curve is not always the same distance from point B on the y-axis
- The problem is asking to calculate the distance when the car is closest to Benny: shortest BP distance

Step 2: Link to mathematical content knowledge, use of concepts (draw inferences & select relevant information)

The curve represents the road and its equation is given Point P represents the car, but we do not know its coordinates Point P is on the curve (car on the road)	$y = x^2 + 2$ $P(x; y)$ $P(x; x^2 + 2)$
The distance between two points, B and P (Benny and car) is unknown	$B\ (0; 3)$ and $P(x; x^2 + 2)$ $PB = \sqrt{(x-0)^2 + (x^2 + 2 - 3)^2}$ $PB = \sqrt{x^2 + (x^2 - 1)^2}$ $PB = \sqrt{x^2 + x^4 - 2x^2 + 1}$ $PB = \sqrt{x^4 - x^2 + 1}$
The distance between two points has to be minimum (shortest distance)	PB will be a minimum if PB^2 is a minimum

Step 3: Completing the task; application of knowledge & use of the correct argument format

To find minimum distance: Step 1: Differentiate the given expression and find the values of x where 1st derivative = 0	$\frac{d(PB^2)}{dx} = 4x^3 - 2x$ $4x^3 - 2x = 0$ $x(2x^2 - 1) = 0$ $x = 0$ or $x^2 = \frac{1}{2}$ $x = \frac{1}{\sqrt{2}}$

Step 2: Substitute the values of x into the distance formula to find the minimum distance.	$PB^2 = \left(\frac{1}{\sqrt{2}}\right)^4 - \left(\frac{1}{\sqrt{2}}\right)^2 + 1$ $= \frac{1}{4} - \frac{1}{2} + 1$ $= \frac{3}{4}$ $PB = \frac{\sqrt{3}}{2} = 0{,}87$
Conclusion	0,87 [units] is the distance between Benny and the car, when the car is closest to Benny.

Step 4: Teacher reflection with specific focus on potential problems in learner's reasoning

(Extract from NSC report & any other observations)
Alternative reasoning could be constructed in Steps 2 and 3, depending on learner's mathematical and cognitive abilities:

- Shortest distance will be where tangent to curve is perpendicular to the line joining P and the curve.
- Finding the equation of the line segment BP and working out the value of y at P.

QUESTION 9: CALCULUS (OPTIMISATION APPLICATION)

Common errors and misconceptions

(a) Many candidates did not see this question as an optimisation question. The most common incorrect answer was that BP = 1 unit. The other common mistake was to take random points for P and to calculate the length of BP.

(b) Few candidates managed to establish that $BP^2 = x^4 - x^2 + 1$ *or* $BP = \sqrt{x^4 - x^2 + 1}$ but could not differentiate these expressions. Candidates did not realise that BP is a minimum if BP^2 is a minimum. The majority of these candidates then used trial and error to establish the minimum distance of BP.

Suggestions for improvement

(a) Optimisation should not only be seen in the context of measurements, learners also need to be exposed to optimisation of functions.

(b) This section of calculus is often taught towards the end of the year and therefore learners do not get enough opportunity to practise. Teachers should ensure that there is enough time for learners to understand the application fully.

2019 November examination

Question 8

After flying a short distance, an insect came to rest on a wall. Thereafter the insect started crawling on the wall. The path that the insect crawled can be described by $h(t) = (t - 6)(-2t^2 + 3t - 6)$, where h is the height (in cm) above the floor and t is the time (in minutes) since the insect started crawling.

8.1 At what height above the floor did the insect start to crawl? (1)

8.2 How many times did the insect reach the floor? (3)

8.3 Determine the maximum height that the insect reached above the floor. (4)

Solution with teaching and learning steps

Step 1: Construct learners' understanding of the problem (extrapolating information/inferencing; developing reasoning)

- An insect is flying and coming to rest before crawling on the wall.
- Curve $h(t)$ represents the crawling path of the insect
- The starting position on the wall is the height h above the floor where insect landed after flying
- We do not know in which direction the insect crawls: towards the floor or towards the roof
- The problem is asking to find: 1) the height above the floor where the insect starts to crawl; 2) the number of times the insect will reach the floor; 3) how high the insect will crawl

Step 2: Link to mathematical content knowledge, use of concepts (draw inferences & select relevant information)

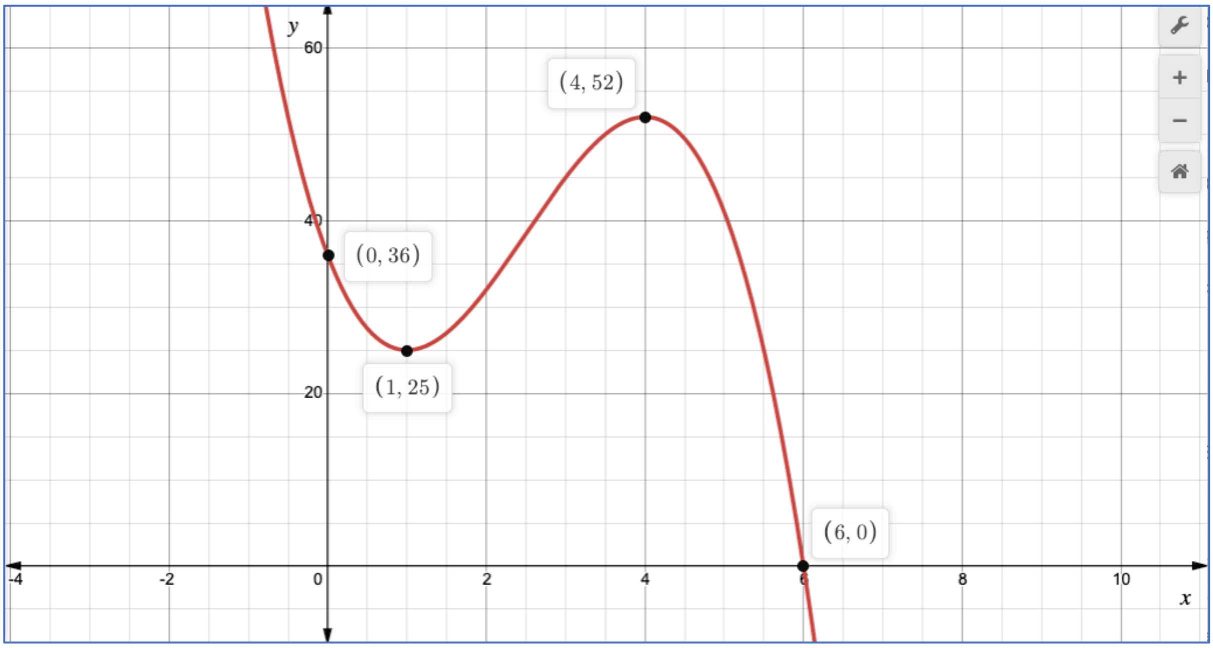
The insect lands on the wall before starting to crawl, which means the time starts at that point. The "crawling" curve's equation is given, it describes the height above the floor at any given time.	$t = 0$ $h(t) = (t - 6)(-2t^2 + 3t - 6)$
The insect could crawl up and down on the wall, never reaching the floor or reaching the floor once or more than once. When the insect reaches the floor, the height above the floor is 0.	$h(t) = 0$
The insect could only crawl as high as the wall is but will not necessarily reach the top of the wall. The maximum height of the insect describes how high the insect will crawl on the wall.	$\max h(t)$ is where $h'(t) = 0$

Step 3: Completing the task; application of knowledge & use of the correct argument format

The initial height is when the time is 0	$h(0) = (0-6)(-2(0)^2 + 3(0) - 6)$ $= -6(-6) = 36$
To answer the question "when will the insect reach the floor" is to find the time when the height is 0.	$h(t) = (t-6)(-2t^2 + 3t - 6) = 0$ $(t-6)(-2t^2 + 3t - 6) = 0$ $t = 6$ or $-2t^2 + 3t - 6 = 0$ $-2t^2 + 3t - 6 = 0$ has no real solutions, so $t = 6$ and the insect will reach the floor once when the time is 6 minutes
To find maximum height: Step 1: Differentiate the given expression and find the values of x where 1st derivative = 0	$h(t) = -2t^3 + 15t^2 - 24t + 36$ $h'(t) = -6t^2 + 30t - 24$ $-6t^2 + 30t - 24 = 0$ $t^2 - 5t + 4 = 0$ $(t-4)(t-1) = 0$ $t = 4$ or $t = 1$
Step 2: Substitute the values of x into the height formula to find the maximum height.	Only $t = 4$ because $h(t)$ reaches maximum value $h = -2(4)^3 + 15(4) - 24(4) + 36$ $= 52\ cm$

Step 4: Teacher reflection with specific focus on potential problems in student's reasoning
(Extract from NSC report & any other observations)

Drawing a curve could assist with visualization and address some of the observed errors & misconceptions:



QUESTION 8: CALCULUS
Common Errors and Misconceptions

a) In Q8.1 many candidates gave the answer as $t = 6$. They calculated the value of t when $h = 0$ instead of calculating the value of h when $t = 0$.

b) Many candidates confused 'how many times' with 'at what time' in answering Q8.2. They gave the answer as $t = 6$ instead of indicating that the insect reached the floor only once.

c) In answering Q8.3 some candidates could not correctly multiply out $h(t)$. Some candidates calculated $h'(t)$, but did not equate $h'(t)$ to 0. Some calculated the values of t but did not calculate the maximum height. Other candidates calculated $h''(t)$, equated it to 0, and solved for t not realizing that in this way they will calculate the t -value of the point of inflection and not the maximum.

Suggestions for Improvement

a) When teaching graphs of cubic functions, teachers should also include those that have one stationary point.

b) Teachers need to expose learners to word problems in order for them to gain confidence.

c) The calculation of critical values should not only be restricted to graphical questions. Expose learners to calculating critical values in contextual questions as well. This will help learners to appreciate the calculations that they perform in Mathematics.

2016 November examination

QUESTION 10

The number of molecules of a certain drug in the bloodstream t hours after it has been taken is represented by the equation $M(t) = -t^3 + 3t^2 + 72t$, $0 < t < 10$.

10.1 Determine the number of molecules of the drug in the bloodstream 3 hours after the drug was taken. (2)

10.2 Determine the rate at which the number of molecules of the drug in the bloodstream is changing at exactly 2 hours after the drug was taken. (3)

10.3 How many hours after taking the drug will the rate at which the number of molecules of the drug in the bloodstream is changing, be a maximum? (3)

[8]

Solution with teaching and learning steps

Step 1: Construct learners' understanding of the problem (extrapolating information/inferencing; developing reasoning)

- The number of molecules of a drug in the bloodstream is given by the equation
- The number of molecules M(t) is changing with the time t changing
- The problem is asking to find:
  - 1) The number of molecules when time = 3 h
  - 2) The (instantaneous) rate of change: number of molecules at 2h
  - 3) The time when the rate of change is maximum

Step 2: Link to mathematical content knowledge, use of concepts (draw inferences & select relevant information)

Use the given equation to find the number of molecules when time is given	$M(t) = -t^3 + 3t^2 + 72t$
To find the rate of change, use the 1st derivative of the given function Exactly 2 hours is the given time	$M'(t) = (-t^3 + 3t^2 + 72t)'$
Looking for the maximum rate of change	max $M'(t)$ is when $M''(t) = 0$

Step 3: Completing the task; application of knowledge & use of the correct argument format

Substitute the time t = 3 hours into the equation to find the number of molecules	$M(t) = -t^3 + 3t^2 + 72t$ $M(3) = -(3)^3 + 3(3)^2 + 72(3)$ $= 216$ 216 molecules
Find 1st derivative and substitute the time t = 2 hours to find the rate of change	$M'(t) = (-t^3 + 3t^2 + 72t)'$ $= -3t^2 + 6t + 72$ $M'(2) = -3(2)^2 + 6(2) + 72 = 72$

Interpret the rate answer: molecules per hour	72 molecules per hour
To find the maximum rate of change: Step 1: Differentiate rate of change and find the values of t where 2nd derivative = 0	$M(t) = -t^3 + 3t^2 + 72t$ $M'(t) = -3t^2 + 6t + 72$ $M''(t) = 0$ $-6t + 6 = 0$ $t = 1$
Step 2: Interpret the answer (referring back to the question)	Maximum rate of change of the number of molecules of the drug in the bloodstream is after 1 hour.

Step 4: Teacher reflection with specific focus on potential problems in student's reasoning
(Extract from NSC report & any other observations)

QUESTION 10:
CALCULUS (APPLICATION)
This question was well answered except for Q10.3.
Common errors and misconceptions
(a) Many candidates did not see the connections amongst rate of change, gradient and derivative. Hence, in Q10.3, it was evident that candidates did not link maximum rate of change to the second derivative being equal to zero.
(b) Many candidates determined $M'(t)$, equated it to zero and then solved for t, as is often required in optimisation questions and not realising that this was not the question in this case in Q10.2.
Suggestions for improvement
(a) Learners should be exposed to the integration of topics across papers.
(b) More emphasis needs to be placed on 'rates of change' and what is meant by this term.
(c) The section on measurement/mensuration is taught in Grade 10 and revision should take place in Grades 11 and 12. Make use of models/teaching aids to assist in the teaching of this section.
(d) Expose learners to examples where they have to differentiate with respect to variables other than x.
(e) This section of Calculus is often taught towards the end of the year and therefore learners do not get enough opportunity to practise. Teachers should ensure that there is enough time for learners to understand the application fully.

Rates of change: optimisation

Learners often experience difficulties in this topic. One of the main reasons for this is their ability to understand the wording of the problem. The difficulty often arises when identifying the quantity that is to be optimised and the quantity that is the constraint and writing down equations for each.

Key words
maximise
minimise
optimisation
constraint

Let's look at the example from ZoomIn Mathematics Gr12 Practice Book: Differential Calculus

Optimisation

Optimisation problems will always ask you to maximise or minimise some quantity described in a word problem (instead of immediately giving you a function to maximise/minimise). For example: "Keeping the area of the rectangle fixed, which rectangle will have the smallest perimeter?"

EXAMPLE

Suppose our rectangle has area 15. The perimeter of a rectangle is given by $P = 2x + 2y$ and our constraint is that the area $A = xy = 15$.

Step 1: Construct learners' understanding of the problem (extrapolating information/inferencing; developing reasoning)

The quantity to be optimised: perimeter of the rectangle

The quantity that is the constraint: the area of the rectangle

Step 2: Link to mathematical content knowledge, use of concepts (draw inferences & select relevant information)

Perimeter of the rectangle: $P = 2x + 2y$ to be minimized
Area of the rectangle: $15 = xy$ is the constraint

We can rewrite the constraint equation: $y = \frac{15}{x}$
Now, substitute the constraint into the perimeter equation:
$P = 2x + 2y$
$= 2x + 2\left(\frac{15}{x}\right)$
$= 2x + 15x^{-1}$

Step 3: Completing the task; application of knowledge & use of the correct argument format

Taking the derivative of P with respect to x:

$D_x[P] = 2 - 15x^{-2}$

Since we need a minimum, we know that the gradient P′ must be equal to zero.

$0 = 2 - 15x^{-2}$

Observe that x cannot be equal to zero since the area would also be zero.

It is safe to multiply both sides by x^2 and simplify:

$0 = 2x^2 - 15$

$x^2 = 15$

$x = \pm\sqrt{15}$

The only logical solution is that $x = \sqrt{15}$ since a rectangle cannot have negative length.

Using either the constraint or perimeter formula, we find that $y = \sqrt{15}$ too.

So, the solution is that a rectangle must in fact have side lengths equal to one another to minimise the perimeter.

Step 4: Teacher reflection with specific focus on potential problems in student's reasoning

(link to any useful observations)

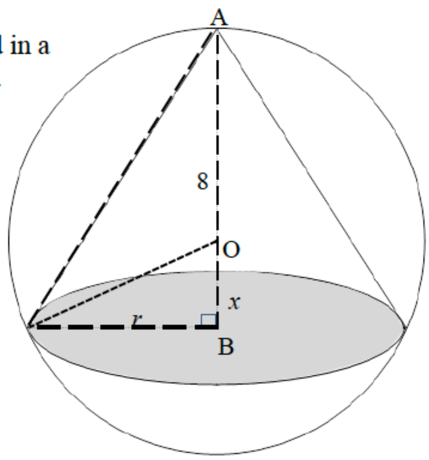
2019 June examination

QUESTION 9

A cone with radius r cm and height AB is inscribed in a sphere with centre O and a radius of 8 cm. OB $= x$.

Volume of sphere $= \frac{4}{3}\pi r^3$

Volume of cone $= \frac{1}{3}\pi r^2 h$



9.1	Calculate the volume of the sphere.	(1)
9.2	Show that $r^2 = 64 - x^2$.	(1)
9.3	Determine the ratio between the largest volume of this cone and the volume of the sphere.	(7)
		[9]

Solution with teaching and learning steps

Step 1: Construct learners' understanding of the problem (extrapolating information/inferencing; developing reasoning)

- A sphere has the radius 8 cm
- A cone is inside the sphere:

- Cone's radius is r cm
- Cone's height is more than the radius of a sphere: $(8 + x)$ cm

- The radius of the sphere is not (*necessarily*!) the same as the radius of the cone
- Volume formulae are given for both shapes
- The problem is asking to find:
  - 1) Volume of the sphere
  - 2) Radius of the cone
  - 3) The ratio between the largest volume of the cone & the volume of the sphere

Step 2: Link to mathematical content knowledge, use of concepts (draw inferences & select relevant information)

Choose the formula to calculate the volume of a sphere with the radius 8 cm	Volume of sphere $= \frac{4}{3}\pi(r)^3$
On the diagram the radius of the cone is r	Volume of cone $= \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi r^2(8 + x)$
There is a right-angled triangle with the base r and other two sides x and 8 (radius of the sphere)	Pythagoras Theorem: $r^2 + x^2 = 8^2$
Looking for largest volume of the cone	$\max V_{cone}$ is where $V'_{cone} = 0$
Ratio of volumes to be found	$\frac{\max V_{cone}}{V_{sphere}} = ?$

Step 3: Completing the task; application of knowledge & use of the correct argument format

Substitute $r = 8$ to calculate the volume of a sphere	Volume of sphere $= \frac{4}{3}\pi(8)^3 = \frac{2048\pi}{3}$ $\approx 2\,144{,}66$
Use Pythagoras theorem to find base r using other two sides x and 8	Pythagoras Theorem: $r^2 + x^2 = 8^2$ $r^2 = 8^2 - x^2$ So, $r^2 = 64 - x^2$ This is the radius of the cone.
To find the largest volume of the cone: Step 1: Differentiate the expression for V_{cone} and find the values of x where 1st derivative = 0 Additional step: substitute radius and height into the formula for V_{cone} and simplify.	Volume of cone $= \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi(64 - x^2)(8 + x)$ $= \frac{\pi}{3}(512 + 64x - 8x^2 - x^3)$ $\frac{dV}{dx} = \frac{64\pi}{3} - \frac{16\pi}{3}x - \frac{3\pi}{3}x^2$ $V'_{cone} = 0$ $0 = 64 - 16x - 3x^2$ $0 = (8 - 3x)(x + 8)$ $x = \frac{8}{3} \qquad x \neq -8$

Step 2: Substitute the values of x into the height formula to find the maximum height.	Volume of cone (max) $= \frac{1}{3}\pi(64 - x^2)(8 + x)$ $= \frac{1}{3}\pi\left(64 - \left(\frac{8}{3}\right)^2\right)\left(8 + \frac{8}{3}\right)$ $= \frac{1}{3}\pi\left(64 - \frac{64}{9}\right)\left(8 + \frac{8}{3}\right)$ $= \frac{1}{3}\pi\left(\frac{512}{9}\right)\left(\frac{32}{3}\right)$ $= \frac{16\,384}{81}\pi$
Set up the ratio of volumes and simplify	$\frac{\frac{16\,384}{81}\pi}{\frac{2048\pi}{3}} = \frac{8}{27} \approx 0{,}3$

Step 4: Teacher reflection with specific focus on potential problems in student's reasoning

(Link to any useful observations)